

18 Systems of first order ODE. Introduction

Up till now we were studying one equation at a time. In this last part of the course we will be studying systems of first order ODE. It is quite a straightforward procedure to rewrite any explicit ODE of the n -th order as a system of n first order equations.

Example 1. Consider the IVP for the pendulum:

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0, \quad \theta(0) = \theta_0, \theta'(0) = v_0.$$

Introduce new variables

$$y_1(t) = \theta(t), \quad y_2 = \theta'(t).$$

We have

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -\frac{g}{l} \sin y_1, \end{aligned}$$

with the initial conditions

$$y_1(0) = \theta_0, \quad y_2(0) = v_0.$$

Therefore, instead of one second order differential equation we end up with a system of two first order equations. It turns out that this particular form is arguably the most convenient way to look at the ordinary differential equations from both theoretical and computational points of view.

Example 2. Rewrite the equation

$$y''' + 5ty'' + 3y = 0$$

as a system of three first order equations. Again, the same approach: Consider new variables:

$$y_1 = y, \quad y_2 = y', \quad y_3 = y''.$$

Therefore,

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_3, \\ \dot{y}_3 &= -5ty_3 - 3y_1, \end{aligned}$$

and we obtain a system of three first order ordinary differential (linear in this case) equations. If together with the original equation we had to solve an IVP, i.e., we also had the initial conditions

$$y(t_0) = a_0, \quad y'(t_0) = a_1, \quad y''(t_0) = a_2,$$

(the number of initial conditions coincides with the order of the equations), then for the system our initial conditions turns into

$$y_1(t_0) = a_0, \quad y_2(t_0) = a_1, \quad y_3(t_0) = a_2,$$

and the number of initial conditions is equal to the number of the variables and the number of equations in the system of first order ODE.

Is it possible to go in the opposite direction — from systems to one equation? In most of the cases the answer is negative, but for linear systems there is a simple approach.

Example 3. Consider the system of two first order ODE:

$$\begin{aligned}\dot{y}_1 &= ay_1 + by_2, \\ \dot{y}_2 &= cy_1 + dy_2,\end{aligned}$$

where $a, b, c, d \in \mathbf{R}$ are some parameters. Differentiate, for example, the first equation:

$$\ddot{y}_1 = a\dot{y}_1 + b\dot{y}_2 = a\dot{y}_1 + b(cy_1 + dy_2).$$

From the first equation we also can find

$$y_2 = \frac{1}{b}(\dot{y}_1 - ay_1),$$

therefore,

$$\ddot{y}_1 = ay_1 + b\left(cy_1 + \frac{d}{b}(\dot{y}_1 - ay_1)\right),$$

which is a linear second order ODE with constant coefficients. To obtain a final answer for the system, we need to present a *pair* of functions as a solution. After finding y_1 from the last equation, you can use the expression for y_2 to write it down. In the final answer there should be two arbitrary constants, which can be present in both expressions for y_1 and y_2 .

Most of the time we will be studying *linear* systems of ODE with *constant coefficients*. To formulate the problem, I will use the matrix notations (more on this in the next lecture). First, we have a square matrix $\mathbf{A} = [a_{ij}]_{n \times n}$, which is given explicitly as

$$\mathbf{A} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}.$$

To denote matrices I usually use capital letters and bold font.

We also have vector-function

$$\mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

and the unknown vector-function

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix},$$

both of which are column vectors with n components. In these lecture notes I use small letters and bold font to point that this is a vector, whereas in the class I usually use the notations $\vec{f}(t)$ and $\vec{y}(t)$. Finally, recall that the derivative of a vector function means the vector of derivatives of its components.

Definition 4. The system

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{f}(t) \tag{1}$$

is called the linear system of n first order ODE with constant coefficients. This system is called homogeneous if $\mathbf{f}(t) \equiv 0$, and nonhomogeneous otherwise. System (1) together with the initial condition

$$\mathbf{y}(t_0) = \mathbf{y}_0 \in \mathbf{R}^n \tag{2}$$

is called the initial value problem (IVP).

Note how the matrix notations allow to write in a concise way. In coordinates system (1) takes the form

$$\begin{aligned} \dot{y}_1 &= a_{11}y_1 + \dots + a_{1n}y_n + f_1(t), \\ &\vdots \\ \dot{y}_n &= a_{n1}y_1 + \dots + a_{nn}y_n + f_n(t). \end{aligned}$$

In most of the cases we will be dealing with systems of two or three first order equations. However, the general theory remains valid for any n .

Example 5. Consider the system

$$\begin{aligned} \dot{y}_1 &= -y_1, \\ \dot{y}_2 &= 2y_2. \end{aligned}$$

Actually, the only thing that makes this pair of equations a *system* is the fact that I announced that this is a system. Otherwise, the equations are not connected to each other, or *decoupled*. This is always true for any linear system, if the matrix is *diagonal*, in this particular case:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

We can solve these equations separately, obtaining

$$\begin{aligned} y_1(t) &= C_1 e^{-t}, \\ y_2(t) &= C_2 e^{2t}. \end{aligned}$$

Of course, we can plot these solutions in the usual manner, in coordinates (t, y_1) and (t, y_2) . Another nice geometric way to look at this solution is to note that the pair $(y_1(t), y_2(t))$ defines a parametric curve on the plane (y_1, y_2) . Moreover, we have a direction on this curve, which is given by the increase of the parameter t , which is often associated with time. Let us figure out what kind of curves on the plane (y_1, y_2) are given by our solution. For this we eliminate t from the solutions, getting

$$\frac{y_1^2}{C_1^2} = e^{-2t} = \frac{C_2}{y_2} \implies y_2 = \frac{C}{y_1^2},$$

which is straightforward to sketch (left as an exercise). The plane (y_1, y_2) is called the *phase plane*, and the parametric curves defined by the solutions of ODE are called *orbits*. One of our main goals is to learn how to figure out the structure of orbits on the phase plane for any linear homogeneous system of two first order ODE.

In the last example we were able to solve our system because the equations were decoupled. It turns out that this is the general approach to the linear homogeneous system: In order to solve a system one needs to find a change of variables such that in the new coordinates the system becomes decoupled (or as simple as possible, if we cannot decouple the equations). Much more on this will be covered in the following lectures.